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The Perfect Numbers and Pascal's Triangle

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The fundamental theorem of arithmetic states that every positive integer can be represented uniquely as the product of prime factors. An integer **n**>1 shall accordingly be written

$$n = p_1^{\alpha 1} p_2^{\alpha 2} \dots p_r^{\alpha r} \qquad (1)$$

where the p_i's are the distinct prime factors and α_i is the multiplicity of p_i (the number of times p_i occurs in the prime factorization).

A positive integer is called a **perfect number** if it is equal to the sum of all its positive divisors other than itself. The sum of divisors of a number **n** with the prime factorization (1) is

$$\sigma(\mathbf{n}) = \frac{p_1^{\alpha 1+1}-1}{p_1-1} \cdot \frac{p_2^{\alpha 2+1}-1}{p_2-1} \cdots \frac{p_r^{\alpha r+1}-1}{p_r-1} = \prod_{i=1}^{r} \frac{p_i^{\alpha i+1}-1}{p_i-1} \qquad (2)$$

The condition for a perfect number may then be given by $\mathbf{n} = \sigma(\mathbf{n}) - \mathbf{n}$ or equivalently, $\sigma(\mathbf{n}) = 2\mathbf{n}$.

Euclid argued that if 2^p-1 is prime for p>1, then

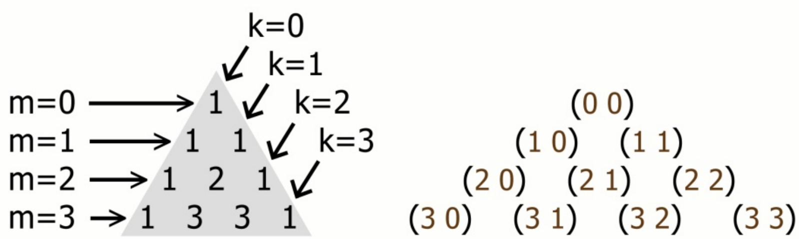
$$P = 2^{p-1}(2^p-1) \qquad (3)$$

is a perfect number. Euler showed later that all even perfect numbers must be of this type (see [4]). The number 2^p-1 is known as a **Mersenne prime** and is denoted by M_p, as in [3]. All perfect numbers known are even and the question of whether there is an odd perfect number is still unanswered. There is no evidence to prove or disprove the existence of an odd perfect number but if one does exist, it must be greater than 10^{100} (see [1]).

For any positive integer m and any integer k satisfying 0<=k<=m, the binomial coefficient (m k) is defined by

$$(m\ k) = \frac{m!}{k!(m-k)!} \tag{4}$$

Use will now be made of the configuration known as Pascal's Triangle in which the binomial coefficient (m k) appears as the (k+1)st number in the (m+1)st row, as in [5].



Integrally | Binomial Coefficient (m k)

Figure 1

The borders of the triangle are composed of ones; a number not on the border is the sum of the two numbers nearest it in the row above.

All even perfect numbers can be shown to lie on the third diagonal of Pascal's Triangle (see Figure 1). The restriction for m is that it must be equal to a Mersenne prime plus one; that is, $m=M_p+1$. Setting k equal to 2 (since the third diagonal of Pascal's triangle is k=2),

$$(m\ k)=\frac{(M_p+1)!}{2!(M_p+1-2)!}=\frac{2^p!}{2!(2^p-2)!}=\frac{2^p(2^p-1)(2^p-2)!}{2!(2^p-2)!}=\frac{2^p(2^p-1)}{2}=2^{p-1}(2^p-1) = P,$$

which is an even perfect number by (3) above.

As in [5], we now note that each number in Pascal's triangle is the sum of the numbers in the preceding diagonal (see Figure 2):

$$\binom{m}{k} = \sum_{i=k-1}^{m-1} \binom{i}{k-1}$$

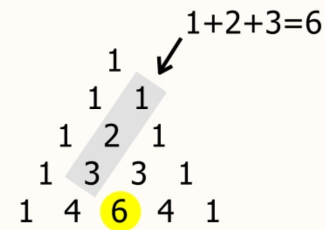


Figure 2

We have seen that all even perfect numbers are on the third diagonal of Pascal's triangle. Hence, the second diagonal would generate the perfect numbers. That is, every even perfect number is the sum of the first $2^p-1=M_p$ (Mersenne prime) numbers:

$$P = \sum_{i=1}^{M_p} i \qquad (5)$$

We now observe that the elements of the third diagonal are the triangular numbers and every even perfect number is triangular in shape (see [2]). (See Figure 3.)

1
1 1
1 1 1

The perfect number 6 with base $M_p=3$

Figure 3

According to Burton [1], there are 24 even perfect numbers known to date (1976). The first 5 and their associated Mersenne primes are given in Table 1 on the next page.

p	M_p	$P=M_p(2^{p}-1)$
2	3	6
3	7	28
5	31	496
7	127	8,128
13	8,191	33,550,336

Table 1

We now have several different ways of computing perfect numbers. We must first compute Mersenne primes M_p. Knowing the Mersenne primes, we can:

(a) compute $P=M_p(2^{p-1})$, using Euclid's formula,

(b) compute $P = \sum_{i=1}^{M_p} i$, summing up the first M_p positive integers, or

(c) with $m=M_p+1$ and $k=2$, compute $P=(m\ k)$.

It is from the last item that we note all even perfect numbers are on the third diagonal of Pascal's triangle.

References

[1] Burton, David M., *Elementary Number Theory*, Allyn and Bacon, Inc., Boston London Sydney, 1976.

[2] Dickson, Leonard Eugene, *History of the Theory of Numbers*, 3 vols., Chelsea Publishing Co., New York, 1952.

[3] Dodge, Clayton W., *Numbers and Mathematics*, Prindle, Weber, and Schmidt, Inc., Boston 1969.

[4] Shanks, Daniel, *Solved and Unsolved Problems in Number Theory*, Spartan Books, Washington, D.C., 1962.

[5] Uspenski, V.A., *Pascal's Triangle*, The University of Chicago Press, Chicago and London, 1974.

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NATIONAL HONORARY MATHEMATICS FRATERNITY

November 15, 1978

Mr. Robert A. Antol
[redacted]

Dear Mr. Antol:

It is a pleasure to send you the enclosed check in recognition of your winning the recent manuscript contest. On behalf of the Council, I extend to you our congratulations and our best wishes for your continuing progress in mathematical endeavors.

Sincerely yours,

RAGood

Richard A. Good
Secretary - Treasurer